Rover Publications
United International Journal of Engineering and Sciences (UIJES)
An International Peer-Reviewed Engineering and Science Journal
www.uijes.com Vol-1,Issue-2 ,2021 ISSN:2582-5887

Oscillation Criteria for Third Order Nonlinear Delay Differential Equations with Damping

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Abstract: This note is concerned with the oscillation of third order non-liner delay differential equation of the form

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)f\left(y(g(t))\right) = 0 \qquad (*)$$

In the papers (A. Tiryaki. M.F Aktas oscillation criteria of a certain class of third order non-liner delay differential equation with damping J .Math Appl 325(2007) 54-68) and (M.f Aktas A. Tiryaki. A. Zafer oscillation criteria for third order non-linear functional differential equation Applied Math. Letter 23 (2010) 756-762)

The Authors established some Sufficient conditions Which insure that any solution of equation $(*)$ oscillator or converges to Zero. provided that the second order equation

$$\left(r_2(t)z'(t)\right)' + \left(p(t)/r_1(t)\right)z(t) = 0 \qquad (**)$$

is non oscillatory. Here we shall improve and unify the result given in the above mentioned papers and present some new sufficient conditions which insure that any solution of equation $(*)$ oscillates equation $(**)$ is non oscillatory we also establish result for the oscillation of equation $(*)$ when equation $(**)$ is oscillatory.

Keywords: oscillation, third order delay differential equation.

1. INTRODUCTION

In this chapter, we consider a nonlinear third order functional differential equations of the form

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)f\left(y(g(t))\right) = 0 \qquad (1)$$

Where $p,q,r_2 \in C(I,\Re)$, $r_1 \in C^2(I,\Re)$, $I=[t_0,\infty)\subset \Re$, $t_0 \ge 0$, $r_1(t)>0$, $r_2(t).>0$, $p(t)\ge 0$, $q(t)>0$, $g \in C^1(I,\Re)$ satisfies $0<g(t)\le t$, $g^1(t)\ge 0$ and $g(t)\to\infty$ as $t\to\infty$ and $f\in C(\Re)$ satisfies $\frac{f(u)}{u}\ge K>0$ for some constant k and $u\ne 0$.

A function $y(t)$ is called a solution of equation (1) if $y(t)\in C[t_y,\infty)$, $r_1(t)y'(t)\in C'[t_y,\infty)$ and $r_2(t)(r_1(t)y'(t))' \in C^2[t_y,\infty)$ and $y(t)$ satisfies equation (1) on $[t_y,\infty)$ for every $t\ge t_y\ge t_0$.

We restrict our attention to those solutions of equation (1) which exist on I and satisfy the condition $\sup\{|y(t)|: t_1\le t<\infty\}>0$. Such a solution is called oscillatory if it has arbitrarily large number of zeros, otherwise it is called non oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Determining oscillation criteria for particular second order differential equations has received a great deal of attention in the last few years. Compared to second order differential equations, the study of oscillation and asymptotic behavior of third order differential equations has received considerably less attention in the literature. In the ordinary case for some recent results on third order equation the reader can refer to Cecchi and Marini $[3,4]$, Parhi and Das $[10,11]$, Parhi and Padhi $[12]$, Skerlik $[13]$, Tiryaki and Yaman $[14]$, Aktas and Tiryaki $[1]$. It is interesting to note that there are third order delay differential equations which have only oscillatory solution or have both oscillatory and non oscillatory solution. For example, $y'''(t)+2y'(t)+y(t-\pi/2)=0$ admits an oscillatory solution $y_1(t)=\sin t$ and a non oscillatory solution $y_2(t)=e^{\lambda t}$, where $\lambda<0$ is a root of the characteristic equation

$$\lambda^3+2\lambda+e^{-\lambda\pi/2}=0.$$

On the other hand, all solutions of

$$y'''(t)+y(t-\tau)=0,\ \tau>0,$$

are oscillatory if and only if $\tau e>3$. But the corresponding ordinary differential equation

$$y'''(t)+y(t)=0,$$

admits a non oscillatory solution $y_1(t)=e^{-t}$ and oscillatory solutions

$$y_2(t) = e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \text{ and } y_3(t) = e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right).$$

In the literature there are some papers and books, for example Agarwal et al. [2], Grace and Lalli [5], Parhi and Das [10,11], Parhi and Padhi [12], Skerlik [13], and Tiryaki and Yaman [14], which deal with the oscillatory and asymptotic behaviour of solutions of functional differential equations. In [1,15], the authors used a generalized Riccati transformation and an integral averaging technique for establishing some sufficient conditions which insure that any solution of equation (1) oscillates or converges to zero. The purpose of our study is to improve and unify the results in [1,15] and present some new sufficient conditions which insure that any solution of equation (1) converges to zero, when the equation

$$\left(r_2(t)z'(t)\right)' + \frac{\rho(t)}{r_1(t)}z(t) = 0$$

Is non oscillatory.

We also apply our results to the equations of the form

$$a_3(t)y'''(t) + a_2(t)y''(t) + a_1(t)y'(t) + q^*(t)f\left(x(g(t))\right) = 0, \qquad (2)$$

Where $a_i(t), i = 1,2,3$ and $q^*(t)$ are positive continuous functions on $[t_0, \infty)$, g and f are as in equation (1).

2 .MAIN RESULTS

For the sake of brevity, we define

$$L_0 y(t) = y(t),\ L_i y(t) = r_i(t)\left(L_{i-1}y(t)\right)',\ i = 1,2 \text{ and } L_3 y(t) = \left(L_2 y(t)\right)'$$

For $t \in [t_0, \infty)$. So equation (1) can be written as

$$L_3 y(t) + p(t)y'(t) + q(t)f\left(y(g(t))\right) = 0.$$

Remark 2.1 If y is a solution of (1) then $z = -y$ is also a solution of the equation

$$L_3 z(t) + p(t)z'(t) + q(t)f^*(z(g(t))) = 0,$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \neq 0$. Thus, concerning non oscillatory solutions of (1) we can restrict our attention only to solutions which are positive for all large t.

Define the functions

$$R_1(t,t_1) = \int_{t_1}^{t} \frac{ds}{r_1(s)} \qquad \text{and} \qquad R_2(t,t_1) = \int_{t_1}^{t} \frac{ds}{r_2(s)}$$

For $t_0 \le t_1 \le t < \infty$. we assume that

$$R_1(t,t_0) \to \infty \quad \text{as}\ t \to \infty, \tag{3}$$

and

$$R_2(t,t_0) \to \infty \quad \text{as}\ t \to \infty. \tag{4}$$

In this section we state and prove the following lemmas which we will use in the proof of our main results.

Lemma 2.2 Suppose that

$$(r_2(t)z'(t))' + \left({p(t)}/{r_1(t)} \right) z(t) = 0 \tag{5}$$

is non-oscillatory. If y is a non-oscillatory solution of (1) on $[t_1, \infty)$, $t_1 \ge t_0$ then there exists a $t_2 \in [t_1, \infty)$ such that $y(t)L_1(y(t)) > 0$ or $y(t)L_1(y(t)) < 0$ for $t \ge t_2$.

In the following two Lemmas, we consider the second order delay differential equation

$$(r_2(t)x'(t))' = Q(t)x(h(t)), \tag{6}$$

Where $r_2(t)$ is as in equation (1), $Q \in C(I, \Re)$, and $h \in C^1(I, \Re)$ such that $h(t) \le t,\ h'(t \ge 0)$ for $t \ge t_0$ and $h(t) \to \infty$ as $t \to \infty$.

Lemma 2.3 If

$$\limsup_{t\to\infty} \int_{h(t)}^{t} Q(s)R_2(h(t),h(s))ds > 1, \qquad (7)$$

then all bounded solutions of equation (6) are oscillatory.

Proof. Let $x(t)$ be a bounded non-oscillatory solution of equation (6), say, $x(t) > 0$ and $x(h(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Then there exists a $t_2 \ge t_1$ such that

$$x(t) > 0,\ x'(t) < 0 \text{ and } (r_2(t)x'(t))' \ge 0 \text{ for } t \ge t_2. \qquad (8)$$

Otherwise, $x'(t) > 0$ for $t \ge t_1$ and so, there exists a constant $c^* > 0$ and a $t_1^* \ge t_1$ such that

$$r_2(t)x'(t) \ge c^* \text{ for } t \ge t_1^*.$$

Integrating this inequality from t_1^* to t and using condition (4) we see that $x(t) \to \infty$ as $t \to \infty$, which contradicts the fact that $x(t)$ is bounded on $[t_1, \infty)$. Now for $v \ge u \ge t_2$ we have

$$x(u) - x(v) = -\int_u^v x'(s)ds$$

$$= -\int_u^v (r_2(s))^{-1}(r_2(s)x'(s))ds$$

$$\ge \left(\int_u^v (r_2(s))^{-1} ds\right)(-r_2(v)x'(v))$$

$$= R_2(v,u)(-r_2(v)x'(v)). \qquad (9)$$

For $t \ge s \ge t_2$, setting $u = h(s)$ and $v = h(t)$ in (9), we get

$$x(h(s)) \ge R_2(h(t),h(s))(-r_2(h(t))x'(h(t))). \qquad (10)$$

Integrating equation (6) from $h(t) \ge t_2$ to t, we have

$$-r_2(h(t))x'(h(t)) \ge r_2(t)x'(t) - r_2(h(t))x'(h(t)) = \int_{h(t)}^{t} Q(s)x(h(s))ds. \qquad (11)$$

Using (10) in (11), we have

$$-r_2(h(t))x'(h(t)) \geq \left(\int_{h(t)}^{t} Q(s)R_2(h(t),h(s))ds \right)\left(-r_2(h(t))x'(h(t))\right)$$

or

$$1 \geq \int_{h(t)}^{t} Q(s)R_2(h(t),h(s))ds. \qquad (12)$$

We take the $\limsup$ as $t \to \infty$ of both sides of inequality (12), we have a contradiction to condition (7) and this completes the proof of the lemma.

Lemma 2.4 If

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \left(r_2^{-1}(u) \int_{u}^{t} Q(s)ds \right) du > 1, \qquad (13)$$

then all bounded solutions of equation (6) are oscillatory.

Proof. Let $x(t)$ be a bounded non oscillatory solution of equation (6), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in Lemma 2.3, we obtain (8). Integrating equation (6) from u to t, we have

$$r_2(t)x'(t) - r_2(u)x'(u) = \int_{u}^{t} Q(s)x(h(s))ds$$

or

$$-x'(u) \geq \left(\left(r_2^{-1}(u)\right) \int_{u}^{t} Q(s)ds \right) x(h(t)).$$

Integrating this inequality from $h(t)$ to t, we get

$$x(h(t)) \geq \left[\int_{h(t)}^{t} \left(\left(r_2^{-1}(u)\right) \int_{u}^{t} Q(s)ds \right) du \right] x(h(t))$$

or

$$1 \geq \left[\int_{h(t)}^{t} \left(\left(r_2^{-1}(u) \right) \int_{u}^{t} Q(s)ds \right) du \right]$$

The rest of the proof is similar to that of Lemma 2.3 and hence is omitted. This completes the proof

Now, we are ready to establish main results of this chapter.

Theorem 2.5 Let conditions (3) and (4) hold and equation (5) is non oscillatory. If there exist two functions ρ and $h \in C^1(I,R)$ such that $g(t) \leq h(t) < t, h'(t) \geq 0$ and $\rho(t) > 0$ for $t \geq t_0$ such that

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1(g(s))(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s),t_1))^2}{4\rho(s)R_2(g(s),t_1)g'(s)r_1^2(s)} \right] ds = \infty, \qquad (14)$$

for all large t and condition (7) or (13) holds with

$$Q(t) = [Kq(t)R_1(h(t),g(t)) - (p(t)/r_1(t))] \geq 0 \text{ for } t \geq t_1,$$

Then equation (1) is oscillatory.

Proof. Let $y(t)$ be a non oscillatory solution of (1) on $[t_1,\infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$.

It follows from Lemma 2.2 that $L_1y(t) < 0$ or $L_1y(t) > 0$ for $t > t_1$. If $L_1y(t) > 0$ for $t \geq t_1$, then one can easily see that $L_2y(t) > 0$ for $t \geq t_1$. Otherwise, $L_2y(t) < 0$ for $t \geq t_1$, so there exists a constant $c^* < 0$ and a $t_1^* \geq t_1$ such that

$$L_1y(t) \leq \frac{c^*}{r_2(t)} \text{ for } t \geq t_1^*.$$

Integrating this inequality from t_1^* to t and using condition (4) we see that $L_1y(t) \to -\infty$ as $t \to \infty$. Thus there exists a constant $c^{**} < 0$ and a $t_1^{**} \geq t_1^*$ such that

$$y'(t) \le \frac{c^{**}}{r_1(t)} \text{ for } t \ge t_1^{**}.$$

Integrating this inequality from t_1^* to t and using condition (3) we find that $y(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $y(t) > 0$ for $t \ge t_1$

Next, we define

$$w(t) = \rho(t)\frac{L_2 y(t)}{y(g(t))} \qquad \text{for } t \ge t_1.$$

First we claim that

$$L_1 y(t) \ge L_1 y(g(t)) \ge R_2(g(t), t_1) L_2(y(g(t))) \ge R_2(g(t), t_1) L_2(y(t)) \text{ for } t \ge t_1 \qquad (15)$$

To this end we have,

$$L_1 y(g(t)) \ge \int_{t_1}^{g(t)} \left(L_1 y(s)'\right) ds = \int_{t_1}^{g(t)} \frac{1}{r_2(s)} L_2 y(s) ds \ge L_2 y(g(t)) R_2(g(t), t_1).$$

Since $L_3 y(t) \le 0$, we get $L_2 y(g(t)) \ge L_2 y(t)$.

This completes the proof of the claim.

By (1) and (15), we have

$$w'(t) \le -K\rho(t)q(t) - \left[w^2(t)\left(\frac{R_2(g(t), t_1) g'(t)}{r_1(g(t))\rho(t)}\right) - w(t)\frac{\rho'(t)}{\rho(t)} - \rho(t)\frac{R_2(g(t), t_1)}{r_1(t)} \right] \qquad (16)$$

and hence

$$w'(t) \le -K\rho(t)q(t) + \frac{r_1(g(t))\left(\rho'(t) r_1(t) - \rho'(t) p(t) R_2(g(t), t_1)\right)^2}{4\rho(t) R_2(g(t), t_1) g'(t) r_1^2(t)}$$

Integrating this inequality from t_1 to t we have

$$\int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1(g(s))\left(\rho'(s) r_1(s) - \rho(s) p(s) R_2(g(s), t_1)\right)^2}{4\rho(s) R_2(g(s), t_1) g'(s) r_1^2(s)} \right] ds \le w(t_1) - w(t) \le w(t_1)$$

which contradicts condition (4) Next, we let $L_1 y(t) < 0$ for $t \geq t_1$ and consider the function $L_2 y(t)$. The case $L_2 y(t) \leq 0$ cannot hold for all large t, say $t \geq t_2 \geq t_1$, since by integration of inequality

$$y'(t) \leq \frac{L_2 y(t_2)}{r_1(t)},\ t \geq t_1,$$

we obtain from (3) $y(t) < 0$ for all large t, a contradiction.

Let $y(t) > 0,\ L_1 y(t) < 0$ and $L_2 y(t) \geq 0$ for all large t, say $t \geq t_3 \geq t_2$. Now, for $v \geq u \geq t_3$, we have

$$y(u) - y(v) = -\int_u^v \frac{1}{r_1(\tau)}\left(r_1(\tau) y'(\tau)\right) d\tau \geq \left(\int_u^v \frac{1}{r_1(\tau)} d\tau\right)\left(-L_1 y(v)\right) = R_1(v,u)\left(-L_1 y(v)\right).$$

Setting $u = g(t)$ and $v = h(t)$, we get

$$y(g(t)) \geq R_1(h(t), g(t))\left(-L_1 y(h(t))\right) = R_1(h(t), g(t)) x(h(t)) \text{ for } t > t_{3,}$$

Where $x(t) = -L_1 y(t) > 0$ for $t \geq t_3$. From equation (1) and the fact that x is decreasing and $g(t) \leq h(t) \leq t$ we obtain

$$\left(r_2(t) x'(t)\right)' + \left(p(t)/r_1(t)\right) x(h(t)) \geq K q(t) R_1(h(t), g(t)) x(h(t))$$

or

$$\left(r_2(t) x'(t)\right)' \geq \left(K q(t) R_1(h(t), g(t)) - \left(p(t)/r_1(t)\right)\right) x(h(t)) \text{ for } t \geq t_3.$$

Proceeding exactly as in the proof Lemma 2.3 and Lemma 2.4, we obtain the desired conclusion completing the proof of the theorem.

Remark 2.6 From the proof of Theorem 2.5 we obtain.

$$w'(t) \leq -K\rho(t) q(t) + \frac{r_1(g(t))\left(P(t,t_1)\right)^2}{4\rho(t) R_2(g(t), t_1) g'(t) r_1^2(t)},$$

where $p(t,t_1)=\rho'(t)r_1(t)-\rho(t)p(t)R_2(g(t),t_1)$. Now, if $P(t,t_1)\geq 0$ for $t\geq t_3$,

we have

$$\rho'(t)r_1(t)\geq P(t,t_1)\text{ for } t\geq t_3,$$

and hence

$$w'(t)\leq -K\rho(t)q(t)+\frac{r_1(g(t))(\rho'(t)r_1(t))^2}{4\rho(t)R_2(g(t),t_1)g'(t)r_1^2(t)}\text{ for } t\geq t_3.$$

It is easy to see that condition (14) can be replaced by

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[K\rho(s)q(s)-\frac{r_1(g(s))(\rho'(s)r_1(s))^2}{4\rho(s)R_2(g(s),t_1)g'(s)r_1^2(s)}\right]ds=\infty$$

for all large t.

Next, if the function $P(t,t_1)\leq 0$ for $t\geq t_3$. We see that condition (14) can be replaced by

$$\int_{t_1}^{\infty}\rho(s)q(s)ds=\infty,$$

for all large t.

Finally, If $\rho'(t)\leq 0$ for $t\geq t_3$, we see from (16) that

$$w'(t)\leq -K\rho(t)q(t)+\rho(t)\frac{R_2(g(t),t_1)}{r_1(t)},$$

and so, condition (14) can be replaced by

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[K\rho(s)q(s)-\rho(s)\frac{R_2(g(s),t_1)}{r_1(s)}\right]ds=\infty,$$

for all large t. The details are left.

The following examples are illustrative.

Example 2.7 Consider the equation

$$y'''(t)+e^{-t}y'(t)+\left(1-e^{t}\right)y\left(t-\frac{5\pi}{2}\right)=0. \qquad (17)$$

It is easy to check that all conditions of Theorem 2.5 are satisfied for

$h(t)=t-2\pi$, $K=1$ and $\rho(t)=1$ and hence equation (17) is oscillatory.

One Such solution is $y(t)=\sin t$.

Example 2.8 Consider the equation

$$y'''(t)+e^{2-2t}y'(t)+\frac{1}{e}y(t-1)\left(1+y^{2}(t-1)\right)=0. \quad (18)$$

Here we take $K=1$, $\rho(t)=1$ and $h(t)=t-1/2$. Now, it is easy to check that all hypotheses of Theorem 2.5 are fulfilled except conditions (7) and (13). We note that equation (18) admits a non-oscillatory solution $y(t)=e^{-t}$.

Next, we present the following comparison results.

Theorem 2.9 If in Theorem 2.5 condition (14) is replaced by the first order delay difference equation

$$w'(t)+\left[\frac{p(t)}{r_1(t)}R_2(g(t),t_1)+Kq(t)\left(\int_{t_1}^{t}\frac{R_2(g(s),t_1)}{r_1(s)}ds\right)\right]w(g(t))=0, \qquad (19)$$

and is oscillatory, then the conclusion of Theorem 2.5 holds.

Proof. Let $y(t)$ be a non-oscillatory solution of (1) on $[t_1,\infty)$, $t\geq t_1$. Without loss of generality, we may assume that $y(t)>0$ and $y(g(t))>0$ for $t>t_1$ for some $t_1\geq t_0$. It follows from Lemma 2.2 that $L_1y(t)<0$ or $L_1y(t)>0$ for $t\geq t_1$. If $L_1y(t)>0$ for $t\geq t_1$, then one can easily see that $L_2y(t)>0$ for $t\geq t_1$. As in the proof of Theorem 2.5 we obtain (15).

Form (15), we have

$$r_1(t)y'(t)=L_1y(t)\geq R_2(g(t),t)L_2(y(g(t)))\text{for } t\geq t_1.$$

Dividing this inequality by $r_1(t)$ and integrating from t_1 to t one can easily find

$$y(t)\geq\left(\int_{t_1}^{t}\frac{R_2(g(s),t_1)}{r_1(s)}ds\right)L_2y(g(t)). \quad (20)$$

Using (15) and (20) in equation (1) we have

$$w'(t)+\left(\frac{p(t)}{r_1(t)}\right)R_2(g(t),t)w(g(t))+Kq(t)\left(\int_{t_1}^{t}\frac{R_2(g(s),t_1)}{r_1(s)}ds\right)w(g(t))\leq 0,$$

Where $w(t)=L_2y(t)>0$. This inequality has a positive solution and hence by Theorem 6.3 in $[5]$, equation (19) has a positive solution, which is a contradiction. The proof of the case when $L_1y(t)<0$ for $t\geq t_1$ is similar to that of Theorem 2.5 and hence is omitted. This completes the proof.

The following result is immediate.

Corollary 2.10 If The Theorem 2.5 the condition (14) is replaced by

$$\liminf_{t\to\infty}\int_{g(t)}^{t}\left[\frac{p(u)}{r_1(u)}R_2(g(u),t_1)+Kq(u)\left(\int_{t_1}^{u}\frac{R_2(g(s),t)}{r_1(s)}ds\right)\right]du\geq\frac{1}{e},$$

then the conclusion of Theorem 2.5 holds.

Next, if equation (3) is oscillatory, we give the following result.

Theorem 2.11 Let conditions (3) and (4) hold and equation (5) is oscillatory. If there exists a function $h\in C(\Re)$ such that $g(t)\leq h(t)\leq t$ and $h'(t)\geq 0$ for $t\geq t_0$ such that (7) or (13) holds with $Q(t)$ as in Theorem 2.5, then every solution y of equation (1) is either $y(t)$ is oscillatory or $y'(t)$ is oscillatory.

Proof. Let $y(t)$ be a non oscillatory solution of (1) on $[t_1, \infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ some $t_1 \geq t_0$. Now, we consider the case $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$. If $L_1 y(t) > 0$ for $t \geq t_1$ holds, then equation (1) becomes

$$(r_2(t)x'(t))' + (p(t)/r_1(t))x(t) \leq 0 \text{ for } t \geq t_2 \geq t_1,$$

where $x(t) = L_1 y(t) > 0$. By $[31]$ equation (5) has a positive solution, a contradiction. Proof of the case when $L_1 y(t) < 0$ for $t \geq t_2 \geq t_1$ is similar to that of Theorem 2.5 and hence is omitted. This completes the proof.

Example 2.12 Consider the equation

$$y'''(t) + \frac{1}{2}y'(t) + \frac{1}{2}y\left(t - \frac{3\pi}{2}\right) = 0. \tag{21}$$

Let $h(t) = t - \pi$. It is easy to check that all hypotheses of Theorem 2.9 are satisfied and hence every solution y of equation (21) is oscillatory or y' is oscillatory. One such solution is $y(t) = \sin t$. We note that none of the results in $[3,8,10,11,12,13,14,15]$ are applicable to equation (21).

Finally, we can easily extend Theorems 2.5 and 2.9 to the equation

$$\left(r_2(t)(r_1(t)y'(t))'\right)' + p(t)y'(h(t)) + q(t)f(y(g(t))) = 0, \tag{22}$$

where $h \in C(I, \Re)$ such that $g(t) \leq h(t) \leq t$ and $h'(t) \geq 0$ for $t \geq t_0$.

Theorem 2.13 Let conditions (3) and (4) hold and the equation

$$(r_2(t)x'(t))' + (p(t)/r_1(h(t)))x(h(t)) = 0 \tag{23}$$

is oscillatory. If condition (7) or (13) holds with

$$Q(t) = [Kq(t)R_1(h(t), g(t)) - (p(t)/r_1(h(t)))] \geq 0 \text{ for } t \geq t_1,$$

then every solution y of equation (22) is either $y(t)$ is oscillatory or $y'(t)$ is oscillatory.

Proof. Let $y(t)$ be a non oscillatory solution of (22) on $[t_1,\infty)$, $t \ge t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$.

As in the proof of Theorem 2.5, we obtain either $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \ge t_1$.

If $L_1 y(t) > 0$ for $t \ge t_1$ holds, then equation (22) becomes

$$(r_2(t)x'(t))' + (p(t)/r_1(h(t)))x(h(t)) \le 0 \text{ for } t \ge t_2 \ge t_1,$$

Where $x(t) = L_1 y(t) > 0$. By $[6]$ equation $[4]$ has a positive solution, a contradiction. Proof of the case when $L_1 y(t) < 0$ for $t \ge t_2 \ge t_1$ is similar to that of Theorem 2.5 and hence is omitted. This completes the proof of the theorem.

We note that there are many criteria in the literature for the oscillation of second order dynamic equations, and so by applying these results to equations (1) and (22), we can obtain many oscillation results which are of similar types to these in $[1.15]$ or else, of different types. The formulations of such results are left.

The following examples are illustrative.

Example 2.14 Consider the equation

$$y'''(t) + y'(t-\pi) + 2y\left(t - \frac{3\pi}{2}\right) = 0. \qquad (24)$$

It is easy to check that all hypotheses of Theorem 2.11 are satisfied with $h(t) = t - 2\pi$ and hence every solution y of equation (24) either $y(t)$ is oscillatory or $y'(t)$ is oscillatory. One such solution is $y(t) = \sin t$. We note that none of the known results appeared in the literature are applicable to this equation because of the delay that appeared in the damping term.

Example 2.15 Consider the equation (24) without delays, namely

$$y'''(t) + y'(t) + 2y(t) = 0 \qquad (25)$$

Equation (25) has a non oscillatory solution $y(t) = e^{-t}$ and $y'(t) = -e^{-t}$ is also non oscillatory. Conditions which involved delays in Theorem 4.2.11 are not fulfilled. The solution set of equation (25) is

$$\{e^{-t}, e^{t/2}\cos\left(\sqrt{7}/2\right)t, e^{t/2}\sin\left(\sqrt{7}/2\right)t\}.$$

We note that the presence of delays in equation (25) generate oscillation.

In order to apply results to equation (2). we can rewrite equation (2) in the form

$$\left(\exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)y''(t)\right)' + \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))y'(t)$$
$$+\exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_2(t)/a_3(t))\left(q^*(t)/a_3(t)\right)f(y(g(t))) = 0$$

In this case, our results are applicable to equation (2) if we let

$$r_1(t) = 1,$$

$$r_2(t) = \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right),$$

$$p(t) = \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))$$

and

$$q(t) = \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))\left(q^*(t)/a_3(t)\right).$$

The formulation of the results as a special case of these obtained above are left.

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